Application of the Exp-Function Method for Solving Some Evolution Equations with Nonlinear Terms of any Orders

Abdelhalim Ebaid

Department of Mathematics, Faculty of Science, Tabuk University, P. O. Box 741, Tabuk 71491, Saudi Arabia

Reprint requests to A. E. E.; Email: halimgamil@yahoo.com

Z. Naturforsch. **65a,** 1039 – 1044 (2010); received August 21, 2009 / revised December 21, 2009

In this paper, suitable transformations and a so-called exp-function method are used to obtain different types of exact solutions for some nonlinear evolution equations with variable coefficients and nonlinear terms of any orders. The Korteweg-de Vries equation and the Burgers equation with nonlinear terms of any orders are chosen to show how to apply the exp-function method for these kinds of nonlinear equations. These exact solutions are in full agreement with the previous results obtained by Ebaid and by Zhu.

Key words: Exp-Function Method; Generalized Burgers Equation.

1. Introduction

Recently, He's exp-function method [1 – 4] has attracted much attention for solving nonlinear equations of mathematical physics. The method has been used by many authors to obtain travelling and non-travelling wave solutions, compact-like solutions and periodic solutions of various nonlinear wave equations [5 – 16]. In [6] we showed that the main advantage of this method over the other methods [17 – 28] is that it can be applied to a wide class of nonlinear evolution equations including those in which the odd and even-order derivative terms are coexist. More recently, we applied the exp-function method to obtain different types of exact solutions for the generalized Klein-Gordon equation [7]. In [29, 30] Zhu extended the method to solve differential-difference equations. Also in [31] Zhou et al. suggested a modified version of the exp-function method for nonlinear equations with a high order of nonlinearity. Many other works have been presented recently by the authors [32 – 38]. In this paper, we aim to show how to apply the exp-function method for solving evolution equations with variable coefficients and nonlinear terms of any orders. The generalized Burgers and Korteweg-de Vries (KdV) equations with variable coefficients are chosen to illustrate the method of solution. The two equations are given as:

$$u_t + \alpha(t)u^{\gamma}u_x + \beta(t)u_{xx} = 0 \text{ (generalized Burgers equation with variable coefficients)}, \quad (1)$$

and

$$u_t + \delta(t)u^{\lambda}u_x + \nu(t)u_{xxx} = 0 \text{ (generalized KdV equation with variable coefficients)}. \quad (2)$$

When $\lambda = 1, 2, 3$, (2) reduces to the classical KdV equation, the modified KdV equation, and the quartic KdV equation, respectively.

2. Method of Solution

In this section we discuss how to solve (1) and (2) by the exp-function method. The method can not be used directly to solve these generalized equations, it requires [7]:

(I) A transformation to convert the given generalized nonlinear partial differential equation (PDE) into a generalized nonlinear ordinary differential equation (ODE).

(II) Another transformation to convert the generalized nonlinear ODE into a new one so that the balancing procedure [1, 2] becomes applicable.

The transformation in (I) is already known and given by [8]:

$$u = u(\eta), \quad \eta = k\,x + \int \sigma(t)\mathrm{d}t, \quad (3)$$

while the transformation in (II) is established by two steps:

(i) Firstly, we obtain a value of n from balancing the generalized nonlinear term with the highest derivative linear term.

(ii) Then we construct the other transformation as $u = v^n$, where n is obtained by step (i).

To make the last two steps as clear as possible, let us consider the generalized Burgers equation given in (1) and balancing the generalized nonlinear term $u^\gamma u_x$ with the highest derivative linear term u_{xx}, we obtain $\gamma n + n + 1 = n + 2$, and this gives $n = 1/\gamma$. So, we use the transformation $u = v^{1/\gamma}$ for the generalized Burgers equation (1). Similarly, we use the transformation $u = v^{2/\lambda}$ for the generalized KdV equation (2). In the next section we show the effectiveness of the exp-function method with the transformations introduced above for solving generalized Burgers and KdV equations with variable coefficients given by (1) and (2).

3. Exact Solutions for the Generalized Burgers Equation with Variable Coefficients

Making the transformation (3), (1) becomes

$$\sigma(t)u' + \alpha(t)ku^\gamma u' + \beta(t)k^2u'' = 0. \tag{4}$$

Now, we can use the transformation

$$u = v^{1/\gamma} \tag{5}$$

directly into the last equation as indicated in the previous section, but we may use it after integrating (4) once and setting the constant of integration equal to zero to obtain

$$\sigma(t)u + \frac{\alpha(t)k}{\gamma+1}u^{\gamma+1} + \beta(t)k^2u' = 0, \quad \gamma \neq -1. \tag{6}$$

Now, using the transformation (5) to this equation, yields

$$\sigma(t)v + \frac{\alpha(t)k}{\gamma+1}v^2 + \frac{\beta(t)k^2}{\gamma}v' = 0, \quad \gamma \neq \{0,-1\}. \tag{7}$$

The exp-function method is based on the assumption that travelling wave solutions can be expressed in the following form [1,2]:

$$\begin{aligned} v(\eta) &= \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} \\ &= \frac{a_c \exp(c\eta) + \ldots + a_{-d}\exp(-d\eta)}{a_p \exp(p\eta) + \ldots + a_{-q}\exp(-q\eta)}, \end{aligned} \tag{8}$$

where c, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. To determine the values of c and p we balance the linear term of highest order in (7), v' with the highest order nonlinear term v^2. Using ansatz (8), we have for the linear term of highest order

$$v' = \frac{c_1 \exp[(p+c)\eta] + \ldots}{c_2 \exp[2p\eta] + \ldots} \tag{9}$$

and

$$v^2 = \frac{c_3 \exp[2c\eta] + \ldots}{c_4 \exp[2p\eta] + \ldots}, \tag{10}$$

where c_i are determined coefficients only for simplicity. Following the balancing procedure [1,2] we balance the highest order of the exp-function in (9) and (10) to obtain $p + c = 2c$, and this gives $p = c$. Similarly, to determine the values of d and q we balance the linear term of lowest order in (7) with the lowest order nonlinear term:

$$v' = \frac{\ldots + d_1 \exp[-(q+d)\eta]}{\ldots + d_2 \exp[-2q\eta]} \tag{11}$$

and

$$v^2 = \frac{\ldots + d_3 \exp[-2d\eta]}{\ldots + d_4 \exp[-2q\eta]}. \tag{12}$$

From (11) and (12) we obtain $-(q+d) = -(2d)$ and this gives $q = d$. The simplest choice for c, d, p, and q is $p = c = 1$ and $q = d = 1$. In [1], He and Wu showed that the final solution does not strongly depends upon the choice of values of c, d, p, and q. Accordingly, ansatz (8) becomes

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{13}$$

Substituting (13) into (7) and by the help of the Mathematica software, we obtain

$$\begin{aligned} \frac{1}{A}\big[&C_{-2}\mathrm{e}^{-2\eta} + C_{-1}\mathrm{e}^{-\eta} \\ &+ C_0 + C_1\mathrm{e}^{\eta} + C_2\mathrm{e}^{2\eta}\big] = 0, \end{aligned} \tag{14}$$

where

$$\begin{aligned} A &= (b_0 + \mathrm{e}^{\eta} + b_{-1}\mathrm{e}^{-\eta})^2, \\ C_{-2} &= \frac{ka_{-1}^2\alpha(t)}{\gamma+1} + a_{-1}b_{-1}\sigma(t), \\ C_{-1} &= \frac{2ka_0a_{-1}\alpha(t)}{\gamma+1} + \frac{k^2a_0b_{-1}\beta(t)}{\gamma} \\ &\quad - \frac{k^2b_0a_{-1}\beta(t)}{\gamma} + a_0b_{-1}\sigma(t) + a_{-1}b_0\sigma(t), \end{aligned}$$

$$C_0 = \frac{ka_0^2\alpha(t)}{\gamma+1} + \frac{2ka_1a_{-1}\alpha(t)}{\gamma+1} - \frac{2k^2a_{-1}\beta(t)}{\gamma} + \frac{2k^2a_1b_{-1}\beta(t)}{\gamma}(a_{-1}+a_1b_{-1}+a_0b_0)\sigma(t),$$

$$C_1 = \frac{2ka_0a_1\alpha(t)}{\gamma+1} - \frac{k^2a_0\beta(t)}{\gamma} + \frac{k^2b_0a_1\beta(t)}{\gamma} + a_0\sigma(t) + a_1b_0\sigma(t),$$

$$C_2 = \frac{ka_1^2\alpha(t)}{\gamma+1} + a_1\sigma(t).$$

Solving the following system of algebraic equations by using Mathematica yields

$$C_{-2}=0,\ C_{-1}=0,\quad C_0=0,\ C_1=0,\ C_2=0, \tag{15}$$

and we obtain the following results:

Case 1.

$$\sigma(t) = \frac{-k^2\beta(t)}{\gamma},\quad a_1 = \frac{k(\gamma+1)\beta(t)}{\gamma\alpha(t)},$$
$$b_{-1} = \frac{\gamma a_0\alpha(t)}{k^2(\gamma+1)^2\beta^2(t)}[k(\gamma+1)b_0\beta(t) - \gamma a_0\alpha(t)],$$
$$a_{-1} = 0, \tag{16}$$

where a_0 and b_0 are free parameters.

Case 2.

$$\sigma(t) = \frac{k^2\beta(t)}{\gamma},\quad a_0 = -\frac{k(\gamma+1)\beta(t)}{\gamma b_0\alpha(t)},$$
$$a_1 = 0,\quad a_{-1}=0,\quad b_{-1}=0, \tag{17}$$

b_0 is a free parameter.

Case 3.

$$\sigma(t) = -\frac{k^2\beta(t)}{\gamma},\quad a_1 = \frac{k(\gamma+1)\beta(t)}{\gamma\alpha(t)},$$
$$a_0 = 0,\quad a_{-1}=0,\quad b_{-1}=0, \tag{18}$$

b_0 is a free parameter.

To compare our results with those obtained in [6], let us use the results of Case 2. In this case the exact solution of the generalized Burgers equation is given by

$$u = \left[\frac{-k(\gamma+1)\beta(t)}{\gamma b_0\alpha(t)(b_0+\mathrm{e}^{\eta})}\right]^{1/\gamma},$$
$$\eta = kx + \frac{k^2}{\gamma}\int\beta(t)\mathrm{d}t. \tag{19}$$

Now, setting $\gamma = 1$, $\alpha(t) = 1$, $\beta(t) = -\nu$ (constant), and $b_0 = 1$, the solution given by (19) reduces to

$$u = \frac{2\nu k}{(1+\mathrm{e}^{\eta})}, = \frac{2\nu k\,\mathrm{e}^{-\eta/2}}{(\mathrm{e}^{-\eta/2}+\mathrm{e}^{\eta/2})},$$
$$\eta = k\,x - k^2\nu t. \tag{20}$$

Using the property $\frac{2\exp(-\eta/2)}{\exp(-\eta/2)+\exp(\eta/2)} = 1-\tanh(\eta/2)$, we obtain from (20) the following exact solution of the classical Burgers equation, $u_t + uu_x - \nu u_{xx} = 0$:

$$u(x,t) = \nu k\left(1-\tanh\left[\frac{k}{2}(x-\nu k\,t)\right]\right). \tag{21}$$

This solution can be found in [6 (Eq. 31)]. When $\nu = 1$ we find that

$$u(x,t) = k\left(1-\tanh\left[\frac{k}{2}(x-k\,t)\right]\right). \tag{22}$$

Replacing k by $K/2$ in (22) gives the solution obtained by the decomposition method [39, (Eq. 38)]:

$$u(x,t) = \frac{K}{2}\left(1-\tanh\left[\frac{K}{4}\left(x-\frac{K}{2}t\right)\right]\right). \tag{23}$$

4. Exact Solutions for the Generalized KdV Equation with Variable Coefficients

The transformation (3) converts (2) into

$$\sigma(t)u' + k\delta(t)u^{\lambda}u' + k^3\nu(t)u''' = 0. \tag{24}$$

Integrating this equation once and setting the constant of integration equal to zero, we get

$$\sigma(t)u + \frac{k\delta(t)}{\lambda+1}u^{\lambda+1} + k^3\nu(t)u'' = 0,\quad \lambda \neq -1. \tag{25}$$

Now, using the transformation

$$u = v^{\frac{2}{\lambda}} \tag{26}$$

directly into (25) yields

$$\sigma(t)v + \frac{k\delta(t)}{\lambda+1}v^4 + \frac{2k^3\nu(t)}{\lambda}\left[\left(\frac{2}{\lambda}-1\right)(v')^2 + vv''\right] = 0, \tag{27}$$
$$\lambda \neq \{0,-1\}.$$

Since there is no linear term in (27) and in order to determine the values of c, d, p, and q, we balance the nonlinear term of highest order vv'' with the nonlinear term v^4. Using the ansatz (8) for the nonlinear terms vv'' and v^4, we obtain

$$vv'' = \frac{c_1 \exp[(3p+2c)\eta] + \ldots}{c_2 \exp[5p\eta] + \ldots} \tag{28}$$

and

$$v^4 = \frac{c_3 \exp[(4c)\eta] + \ldots}{c_4 \exp[4p\eta] + \ldots} = \frac{c_3 \exp[(p+4c)\eta] + \ldots}{c_4 \exp[5p\eta] + \ldots}, \tag{29}$$

where c_i are determined coefficients only for simplicity. Following the balancing procedure, we balance the highest order of exp-function in (28) and (29) to obtain $3p+2c = p+4c$ and this gives $p = c$. To get the values of d and q we balance the lowest order of exp-function in (27)

$$vv'' = \frac{\ldots + d_1 \exp[-(3q+2d)\eta]}{\ldots + d_2 \exp[-5q\eta]} \tag{30}$$

and

$$v^4 = \frac{\ldots + d_3 \exp[-4d\eta]}{\ldots + d_4 \exp[-4q\eta]} = \frac{\ldots + d_3 \exp[-(q+4d)\eta]}{\ldots + d_4 \exp[-5q\eta]}, \tag{31}$$

where d_i are determined coefficients only for simplicity. We obtain from (30) and (31) that $-(3q+2d) = -(q+4d)$, and this gives $q = d$. Here, we also choose, $p = c = 1$ and $q = d = 1$, for simplicity. Now the ansatz (8) becomes

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{32}$$

Substituting (32) into (27) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, and $\sigma(t)$. Solving this system with the aid of Mathematica we obtain

$$\sigma(t) = \frac{-4k^3 \nu(t)}{\lambda^2},\quad a_0 = \pm\frac{2k}{\lambda}\sqrt{\frac{2\nu(t)(\lambda+1)(\lambda+2)b_{-1}}{\delta(t)}},\quad a_1 = 0,\ a_{-1} = 0,\ b_0 = 0, \tag{33}$$

where b_{-1} is a free parameter. Substituting these results into (32), we obtain the generalized solitary solution of (2) as

$$u = \left[\pm\frac{2k}{\lambda}\sqrt{\frac{2\nu(t)(\lambda+1)(\lambda+2)b_{-1}}{\delta(t)}}\frac{1}{e^{\eta} + b_{-1}e^{-\eta}}\right]^{\frac{2}{\lambda}},\quad \eta = k\,x - \frac{4k^3}{\lambda^2}\int \nu(t)\mathrm{d}t. \tag{34}$$

To compare our result with those in open literature, we set $b_{-1} = 1$, $\delta(t) = -6$, $\nu(t) = 1$, $\lambda = 1$ (classical KdV equation) and replacing k with $\frac{K}{2}$, then (34) becomes

$$u(x,t) = \frac{-2K^2 e^{K(x-K^2t)}}{(1+e^{K(x-K^2t)})^2}, \tag{35}$$

which is the travelling wave solution obtained by the decomposition method in [40]. From the generalized solitary solution (34), we can obtain the periodic solutions by replacing k with iK, consequently

$$\begin{aligned} e^{\eta} &= e^{\mathrm{i}\left(Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right)} = \cos\left[Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right] + \mathrm{i}\sin\left[Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right],\\ e^{-\eta} &= e^{-\mathrm{i}\left(Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right)} = \cos\left[Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right] - \mathrm{i}\sin\left[Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right]. \end{aligned} \tag{36}$$

By substituting (36) into (34), it then follows

$$u = \left\{\pm\frac{2K}{\lambda}\sqrt{\frac{2\nu(t)(\lambda+1)(\lambda+2)b_{-1}}{\delta(t)}} \left[(1+b_{-1})\cos\left(Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right) + \mathrm{i}(1-b_{-1})\sin\left(Kx + \frac{4K^3}{\lambda^2}\int \nu(t)\mathrm{d}t\right)\right]^{-1}\right\}^{2/\lambda}. \tag{37}$$

Now, to obtain the periodic solutions, we must eliminate the imaginary part in (37) to give $b_{-1} = 1$. Consequently, the generalized periodic solution of (2) is

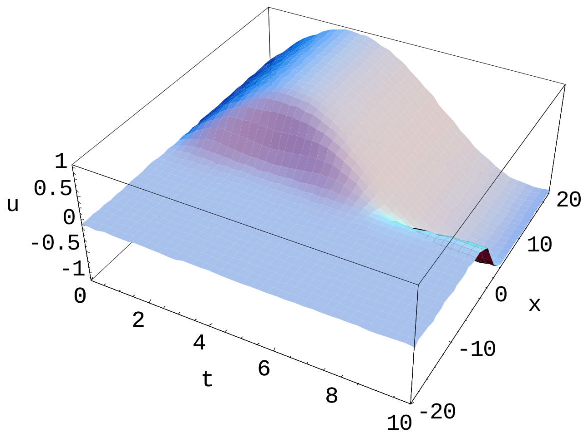

Fig. 1 (colour online). Solution (19) for the Burgers equation with periodic coefficients, where $\gamma = 1$, $\alpha(t) = \cos(0.5t)$, $\beta(t) = \sin(t)$, $b_0 = 2$, and $k = -1$.

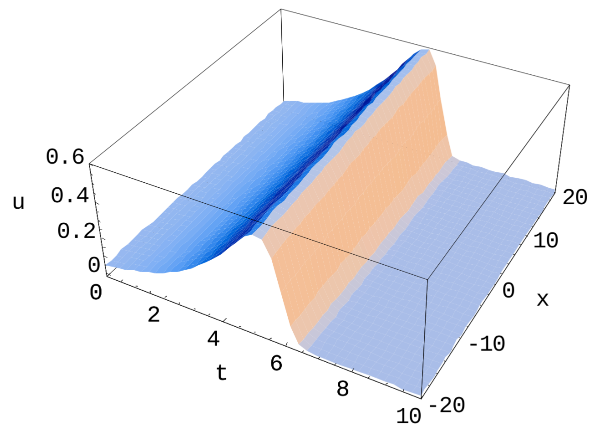

Fig. 2 (colour online). Soliton-like solution (19) for the quadratic Burgers equation ($\gamma = 2$) with exponential coefficients, where $\alpha(t) = e^t$, $\beta(t) = e^{2t}$, $b_0 = 2$, and $k = -0.01$.

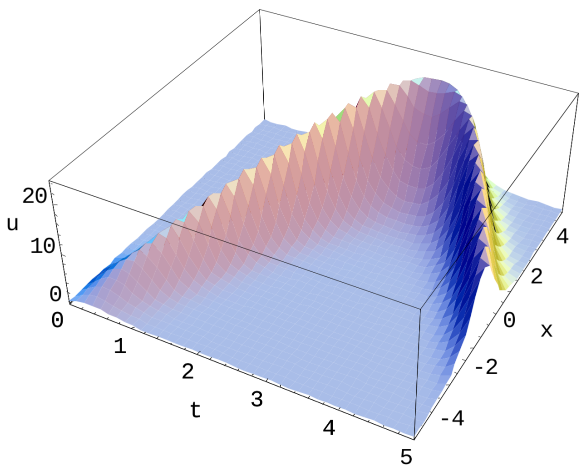

Fig. 3 (colour online). Solution (34) for the kdV equation ($\lambda = 1$) with periodic coefficients, where $\delta(t) = \cos(0.5t)$, $\nu(t) = \sin(t)$, $b_{-1} = 1$, and $k = 1$.

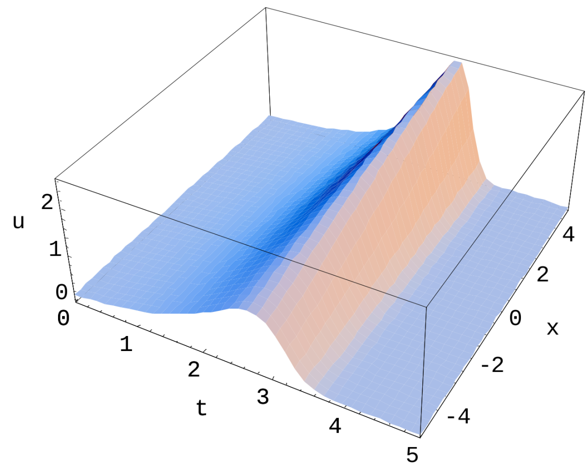

Fig. 4 (colour online). Soliton-like solution (34) for the KdV equation ($\lambda = 1$) with exponential coefficients, where $\delta(t) = e^t$, $\nu(t) = e^{2t}$, $b_{-1} = 0.5$, and $k = 0.1$.

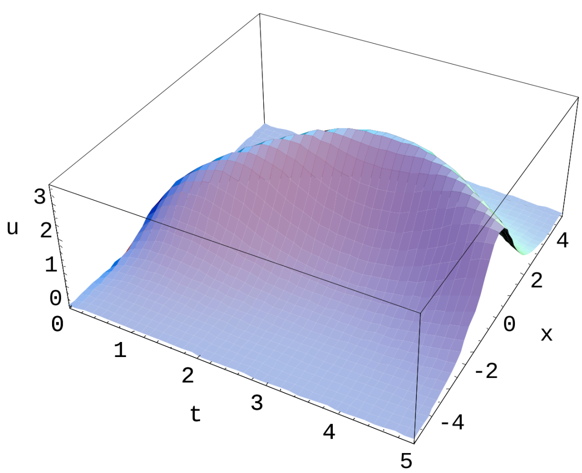

Fig. 5 (colour online). Soliton-like solution (34) for the modified KdV equation ($\lambda = 2$) with periodic coefficients, where $\delta(t) = \cos(0.5t)$, $\nu(t) = \sin(t)$, $b_{-1} = 1$, and $k = 1$.

given by

$$u = \left[\pm \frac{K}{\lambda} \sqrt{\frac{2\nu(t)(\lambda+1)(\lambda+2)}{\delta(t)}} \cdot \frac{1}{\cos\left(Kx + \frac{4K^3}{\lambda^2} \int \nu(t)\mathrm{d}t \right)} \right]^{2/\lambda}, \tag{38}$$

where K is a free real number.

5. Conclusion

Based on the exp-function method, some nonlinear evolution equations with variable coefficients and nonlinear terms of any order are solved exactly. The effectiveness of the method has been tested by applying it to the generalized Burgers equation and the generalized KdV equation. As seen from the example of the generalized Burgers equation, the main advantage of this method over the other methods is that it can be applied to a wide class of nonlinear evolution equations including those in which the odd and even-order derivative terms are coexist. Furthermore, the method leads to both generalized solitary solutions and periodic solutions.

Acknowledgements

The author would like to thank the referees for their comments and discussions.

[1] J.-H. He and X.-H. Wu, Chaos, Solitons, and Fractals **30**, 700 (2006).
[2] X.-H. (Benn) Wu and J.-H. He, Chaos, Solitons, and Fractals **38**, 903 (2008).
[3] X.-H. (Benn) Wu and J.-H. He, Comput. Math. Appl. **54**, 966 (2007).
[4] J.-H. He and L.-N. Zhang, Phys. Lett. A **372**, 1044 (2008).
[5] T. Ozis and C. Koroglu, Phys. Lett. A **372**, 3836 (2008).
[6] A. Ebaid, Phys. Lett. A **365**, 213 (2007).
[7] A. Ebaid, Comput. Appl. Math. **223**, 278 (2009).
[8] S. Zhang, Phys. Lett. A **365**, 448 (2007).
[9] S. Zhang, Chaos, Solitons, and Fractals **38**, 270 (2008).
[10] S. Zhang, Phys. Lett. A **371**, 65 (2007).
[11] S. Zhang, Phys. Lett. A **372**, 1873 (2008).
[12] A. Boz and A. Bekir, Comput. Math. Appl. **56**, 1451 (2008).
[13] E. Yusufoglu, Phys. Lett. A **372**, 442 (2008).
[14] F. Xu, Phys. Lett. A **372**, 252 (2008).
[15] S.-d. Zhu, Phys. Lett. A **372**, 654 (2008).
[16] F. Khani, S. Hamedi-Nezhad, A. Molabahrami, Phys. Lett. A **371**, 234 (2007).
[17] E. J. Parkes and B. R. Duffy, Comput. Phys. Commun. **98**, 288 (1996).
[18] D. S. Li, F. Gao, and H. Q. Zhang, Chaos, Solitons, and Fractals **20**, 1021 (2004).
[19] C. L. Bai and H. Zhao, Chaos, Solitons, and Fractals **27**, 1026 (2006).
[20] H. A. Abdusalam, Int. J. Nonlinear Sci. Numer. Simul. **6**, 99 (2005).
[21] S. Liu, Z. Fu, S. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001).
[22] Z. Fu, S. Liu, S. Liu, and Q. Zhao, Phys. Lett. A **290**, 72 (2001).
[23] G.-T. Liu and T.-Y. Fan, Phys. Lett. A **345**, 161 (2005).
[24] H. Zhang, Commun. Nonlinear Sci. Numer. Simul. **12**, 627 (2007).
[25] E. Fan, Chaos, Solitons, and Fractals **16**, 819 (2003).
[26] J.-L. Zhang, M.-L. Wang, Y.-M. Wang, and Z.-D. Fang, Phys. Lett. A **350**, 103 (2006).
[27] J. Liu and K. Yang Chaos, Solitons, and Fractals **22**, 111 (2004).
[28] H. Zhang, Chaos, Solitons, and Fractals **26**, 921 (2005).
[29] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 461 (2007).
[30] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 465 (2007).
[31] X. W. Zhou, Y. X. Wen, and J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **9**, 301 (2008).
[32] M. A. Abdou and E. M. Abulwafa, Z. Naturforsch. **63a**, 646 (2008).
[33] V. Marinakis, Z. Naturforsch. **63a**, 653 (2008).
[34] S. Zhang, W. Wang, and J. L. Tong, Z. Naturforsch. **63a**, 663 (2008).
[35] H. M. Fu and Z. D. Dai, Int. J. Nonlinear Sci. Numer Simul. **10**, 927 (2009).
[36] W. M. Zhang and L. X. Tian, Int. J. Nonlinear Sci. Numer. Simul. **10**, 711 (2009).
[37] S. Kutluay and A. Esen, Int. J. Nonlinear Sci. Numer Simul. **10**, 717 (2009).
[38] J. H. He and X. U. Fei, Comput. Math. Appl. **58**, 2431 (2009).
[39] T. A. Abassy, M. A. El-Tawil, and H. K. Saleh, Chaos, Solitons, and Fractals **32**, 1008 (2007).
[40] A. M. Wazwaz, Chaos, Solitons, and Fractals **12**, 2283 (2001).